PRESENTATIONS OF SYMBOLIC DYNAMICAL SYSTEMS BY LABELLED DIRECTED GRAPHS (NOTES FOR A "MINI-COURS", SDA2, PARIS 4-5 OCTOBER 2007)

WOLFGANG KRIEGER

ABSTRACT. We develop some aspects of a general theory of presentations of subshifts by labelled directed graphs, in particular by compact graphs. Also considered are synchronization properties of subshifts that lead to presentations by countable graphs.

Let Σ be a finite alphabet. In symbolic dynamics one studies subshifts (X, S_X), X a shift-invariant closed subset of the shift space $\Sigma^{\mathbb{Z}}$ and S_X the restriction of the left shift on $\Sigma^{\mathbb{Z}}$ to X. An introduction to the theory of subshifts is given in [Ki] and in [LM]. See here also [BP].

The first talk is about some aspects of a general theory of presentations of subshifts by labelled directed graphs, in particular by compact graphs. The topic of the the second talk are synchronization properties of subshifts that lead to presentations by countable graphs.

We fix terminology and notation. Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set

$$x_{[i,k]} = (x_j)_{i \leq j \leq k}, \quad x \in X, i, k \in \mathbb{Z}, i \leq k,$$

and

$$X_{[i,k]} = \{x_{[i,k]} : x \in X\}, \quad i, k \in \mathbb{Z}, i \leq k.$$

We use similar notation also for blocks,

$$b_{[i',k']} = (b_j)_{i' \leq j \leq k'}, \quad b \in X_{[i,k]},\ i \leq i' \leq k' \leq k,$$

and also if indices range in semi-infinite intervals. $\mathcal{L}(X)$ is the set of admissible words of X. When convenient we identify blocks with the words they carry. We set

$$\Gamma^+_n(a) = \{b \in X_{(k,k+n]} : (a,b) \in X_{[i,k+n]}\}, \qquad n \in \mathbb{N},$$
$$\Gamma^+_\infty(a) = \{y^+ \in X_{(k,\infty)} : (a, y^+) \in X_{[i,\infty)}\},$$
$$\Gamma^+(a) = \Gamma^+_\infty(a) \cup \bigcup_{n \in \mathbb{N}} \Gamma^+_n(a), \qquad a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k.$$

Γ^- has the time symmetric meaning. We set

$$\omega_n^+(a) = \bigcap_{x^- \in \Gamma_\infty^-(a)} \{b \in X_{(k,k+n]} : (x^-, a, b) \in X_{(-\infty,k+n]}\},$$

$$\omega_\infty^+(a) = \bigcap_{x^- \in \Gamma_\infty^-(a)} \{y^+ \in X_{(k,\infty)} : (x^-, a, y^+) \in X\},$$

$$\omega^+(a) = \omega_\infty^+(a) \cup \bigcup_{n \in \mathbb{N}} \omega_n^+(a), \qquad a \in X_{[i,k]}, i, k \in \mathbb{Z}, i \leq k.$$

ω^- has the time symmetric meaning. The unstable set of a point $x \in X$ is denoted by $W_X^-(x)$,

$$W_X^-(x) = \bigcup_{I \in \mathbb{N}} \{y \in X : y_i = x_i, i \leq I\}.$$

The $W_X^-(x), x \in X$, carry the inductive limit topologies of the compact topologies on the sets $\{y \in X : y_i = x_i, i \leq I\}, I \in \mathbb{N}$.

We recall that, given subshifts $X \subset \Sigma^{\mathbb{Z}}, \bar{X} \subset \bar{\Sigma}^{\mathbb{Z}}$, and a topological conjugacy $\varphi : X \to \bar{X}$, there is for some $L \in \mathbb{Z}_+$ a block mapping

$$\Phi : X_{[-L,L]} \to \bar{\Sigma}$$

such that

$$\varphi(x) = (\Phi(x_{[i-L,i+L]}))_{i \in \mathbb{Z}}.$$

We say then that φ is given by Φ, and we write

$$\Phi(a) = (\Phi(a_{[j-L,j+L]}))_{i+L \leq j \leq k-L}, \quad a \in X_{[i,k]}, \quad i, k \in \mathbb{Z}, k - i \geq 2L,$$

and use similar notation if indices range in semi-infinite intervals. The interval $[-L, L]$ is called a coding window.

1. Presentations of subshifts by Shannon graphs

Let Σ be a finite alphabet and consider a directed graph $\mathcal{G}$ with vertex set $\mathcal{V}$ whose directed edges are labeled with symbols in Σ. The graph $\mathcal{G}$ is called Shannon if its labeling is 1-right resolving in the sense that for every vertex $V \in \mathcal{V}$ and for every symbol $\sigma \in \Sigma$ there is at most one edge in $\mathcal{G}$ that leaves V and that carries the label σ. Shannon graphs are also known as deterministic transition systems. Denote the set of vertices $V \in \mathcal{V}$ of a Shannon graph $\mathcal{G}$ that have an outgoing edge that carries the label $\sigma \in \Sigma$ by $\mathcal{V}(\sigma)$, and for $V \in \mathcal{V}(\sigma)$ denote by $\tau_\sigma(V)$ the final vertex of the edge that leaves V and that carries the label σ. We call $(\tau_\sigma)_{\sigma \in \Sigma}$ the transition rule of the Shannon graph. The forward context $\Gamma_\infty^+(V)$ of a vertex V of a Shannon graph $\mathcal{G}$ is defined as the closed subset of $\Sigma^{\mathbb{N}}$ that contains the label sequences of the semi-infinite paths in $\mathcal{G}$ that leave V. A Shannon graph is called forward separated if distinct vertices have distinct forward contexts.

Call a Shannon graph compact if its vertex set $\mathcal{V}$ carries a compact topology such that the sets $V(\sigma)$, are open and such that the mappings $V \to \tau_\sigma(V), (V \in \mathcal{V}(\sigma)), \sigma \in \Sigma$, are continuous. We say that a Shannon graph $\mathcal{G}$, in which every vertex has at least one incoming edge and at least one outgoing edge presents a subshift $X \subset \Sigma^{\mathbb{Z}}$, if the set of admissible words of X coincides with the set of label sequences of the finite paths in $\mathcal{G}$.

For a finite alphabet Σ denote by $\mathcal{V}(\Sigma)$ the set of non-empty closed subsets of $\Sigma^{\mathbb{N}}$ with its compact Hausdorff subset topology. $\mathcal{V}(\Sigma)$ is the vertex set of a compact Shannon graph $\mathcal{G}(\Sigma)$: For $\sigma \in \Sigma$ the set $\mathcal{V}(\Sigma)(\sigma)$ of $\mathcal{G}(\Sigma)$ is equal to the set of $V \in \mathcal{V}$ that contain a sequence that starts with σ, and the transition rule for $\mathcal{G}(\Sigma)$ is

$$\tau_\sigma(V) = \{v_{(1,\infty)} : v \in V, v_1 = \sigma\}, \qquad V \in \mathcal{V}(\Sigma)(\sigma), \sigma \in \Sigma. \tag{1.1}$$

The sets $\mathcal{V}(\Sigma)(\sigma), \sigma \in \Sigma$, are compact-open and we can associate to the finite alphabet Σ also the topological Markov chain $tM(\Sigma)$ which is the compact dynamical system that is obtained by having the left shift act on the space

$$\{(x_i, V_i)_{i \in \mathbb{Z}} \in (\Sigma \times \mathcal{V}(\Sigma))^{\mathbb{Z}} : V_{i+1} = \tau_{x_{i+1}}(V_i), i \in \mathbb{Z}\}.$$

Denote by $\mathcal{C}(W^-_{\Sigma^{\mathbb{Z}}}(x))$ the set of closed subsets of $W^-_{\Sigma^{\mathbb{Z}}}(x), x \in \Sigma^{\mathbb{Z}}$. For the finite alphabet Σ we introduce the set $\mathcal{C}^\bullet(\Sigma)$ of pairs (x, C), where $x \in \Sigma^{\mathbb{Z}}$ and where $C \in \mathcal{C}(W^-(x))$. Having the left shift act on $\mathcal{C}^\bullet(\Sigma)$ one obtains a dynamical system for which one has a shift commuting bijection ψ_Σ onto $tM(\Sigma)$. This bijection ψ_Σ assigns to a point $(x, C) \in \mathcal{C}^\bullet(\Sigma)$ the point $(x_i, V_i(C))_{i \in \mathbb{Z}} \in tM(\Sigma)$ that is given by

$$V_i(C) = \{y \in C : x_j = y_j, j \le i\}, \qquad i \in \mathbb{Z}.$$

In order to turn $\mathcal{C}^\bullet(\Sigma)$ into a compact dynamical system one transports the topology on $tM(\Sigma)$ to $\mathcal{C}^\bullet(\Sigma)$ by means of the inverse of the bijection ψ_Σ.

Lemma 1. *Let $X \subset \Sigma^{\mathbb{Z}}, \widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be subshifts, and let $\varphi : X \to \widetilde{X}$ be a topological conjugacy. Let $L \in \mathbb{Z}_+$ be such that $[-L, L]$ is a coding window for φ and for φ^{-1}. Let φ be given by the block map $\Phi : X_{[-L,L]} \to \widetilde{\Sigma}$, and let φ^{-1} be given by the block map $\widetilde{\Phi} : \widetilde{X}_{[-L,L]} \to \Sigma$. Let $(x, C) \in \mathcal{C}^\bullet(\Sigma)$, and set $(\widetilde{x}, \widetilde{C}) = \varphi(x, C)$. Then*

$$V_i(\widetilde{C}) = \{\widetilde{\Phi}(y^+) : y^+ \in V_{i-L}(C), \Phi(x_{[i-3L,i-L]}, y^+_{[i-L,L]}) = \widetilde{x}_{[i-2L,i]}\}, \qquad i \in \mathbb{Z}.$$

Proof. One notes that

$$V_i(\widetilde{C}) \supset \{\Phi(x^+_{[i-3L,i+L]}y^+) : y^+ \in V_{i+L}(C)\}, \quad \widetilde{\Phi}(V_i(\widetilde{C}) \subset V_{i-L}(C), \qquad i \in \mathbb{Z}. \quad \square$$

We call a subset $\mathcal{V}$ of $\mathcal{V}(\Sigma)$ transition-complete if for $V \in \mathcal{V} \cap \mathcal{V}(\Sigma)(\sigma)$ also $\tau_\sigma(V) \in \mathcal{V}$. To a transition complete subset $\mathcal{V}$ of $\mathcal{V}(\Sigma)$ we associate the sub-Shannon graph $G(\mathcal{V})$ of $\mathcal{G}(\Sigma)$ that has as vertex set the set $\mathcal{V}$ and as transition rule the restriction of the rule (1.1) to $\mathcal{V}$. The set of forward contexts of the vertices af a forward separated Shannon graph is a transition-complete subset of $\mathcal{V}(\Sigma)$, and the mapping that sends every vertex of a forward separated Shannon graph to its forward context is an isomorphism of $\mathcal{G}$ onto the Shannon graph that is associated with the set of forward contexts of $\mathcal{G}$.

We say that a transition-complete subset $\mathcal{V}$ of $\mathcal{V}(\Sigma)$ is retro-complete if every $V \in \mathcal{V}$ thas a a predecessor in $\mathcal{V}$. For a transition-complete compact subset $\mathcal{V}$ of $\mathcal{V}(\Sigma)$ the sets $\mathcal{V}(\sigma), \sigma \in \Sigma$, are also compact, and we set

$$\tau(\mathcal{V}) = \bigcup_{\sigma \in \Sigma} \tau_\sigma(\mathcal{V}(\sigma)),$$

and with $\tau^{(0)}(\mathcal{V}) = \mathcal{V}$, we set inductively

$$\tau^{(n)}(\mathcal{V}) = \tau(\tau^{(n-1)}(\mathcal{V})), \qquad n \in \mathbb{N}.$$

Here

$$\tau^{(n)}\mathcal{V}) \subset \tau^{(n-1)}(\mathcal{V}), \qquad n \in \mathbb{N},$$

and the intersection $\bigcap_{n \in \mathbb{N}} \tau^{(n)}(\mathcal{V})$ is the maximal transition- and retro-complete subset of $\mathcal{V}$.

The transition- and retro-complete subsets of $\mathcal{V}(\Sigma)$ are in one-to-one correspondence with the shift invariant subsets of $tM(\Sigma)$: To a transition- and retro-complete set $\mathcal{V} \subset \mathcal{V}(\Sigma)$ there corresponds the system

$$tM(\mathcal{V}) = \{(x_i, V_i)_{i \in \mathbb{Z}} \in (\Sigma \times \mathcal{V})^{\mathbb{Z}} : V_{i+1} = \tau_{x_{i+1}}(V_i), i \in \mathbb{Z}\}.$$

Also, assigning to a transition- and retro-complete subset $\mathcal{V}$ of $\{\mathcal{V}(\Sigma)$ the dynamical system $C^\bullet(\mathcal{V}) = \psi_\Sigma^{-1}(tM(\mathcal{V}))$, sets up a one-to one corresondence between the transition- and retro-complete subsets of $\mathcal{V}(\Sigma)$ and the shift invariant subsystems $C^\bullet$ of $C^\bullet(\Sigma)$ with the property that $(x, C) \in C^\bullet$ and $y \in C$ imply hat $(y, C) \in C^\bullet$.

Lemma 2. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift. Let $x^- \in X_{(-\infty,0]}, C \in \mathcal{V}(\Sigma)$, and*

$$x^-(k) \in X_{(-\infty,0]}, \quad C(k) \in \mathcal{V}(\Sigma), C(k) \subset \Gamma^+_\infty(x^-(k)), \qquad k \in \mathbb{N},$$

and let

$$x^- = \lim_{k\to\infty} x^-(k), \quad C = \lim_{k\to\infty} C(k).$$

Then

$$C \subset \Gamma^+_\infty(x^-). \tag{1.2}$$

Proof. For $n \in \mathbb{N}$ let $k_n \in \mathbb{N}$ be such that

$$x^-_{[-n,0]} = x^-_{[-n,0]}(k), \quad C_{[1,n]} = C_{[1,n]}(k), \qquad k \geq k_n.$$

Then for $x^+ \in C$ and $n \in \mathbb{N}$

$$(x^-_{[-n,0]}, x^+_{[1,n]}) = (x^-_{[-n,0]}(k), x^+_{[1,n]}) \in X_{[-n,n]}, \qquad k \geq k_n,$$

which implies (1.2). $\square$

Proposition 1. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift. The set*

$$\mathcal{V}_\circ(X) = \bigcup_{x^-\in X_{(-\infty,0]}} \{V \in \mathcal{V}(\Sigma) : V \subset \Gamma^+_\infty(x^-)\}$$

is closed.

Proof. Apply Lemma 2 and the compactness of $X_{(-\infty,0]}$. $\square$

For a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set

$$\mathcal{V}_{max}(X) = \bigcap_{n\in\mathbb{N}} \tau^n(\mathcal{V}_\circ(X)),$$

$\mathcal{G}(\mathcal{V}_{max}(X))$ presents X and we refer to $\mathcal{G}(\mathcal{V}_{max}(X))$ as the maximal presenting Shannon graph of X. This terminology is justified by the fact that every transition- and retro-complete subset $\mathcal{V}$ of $\mathcal{V}(\Sigma)$, such that $\mathcal{G}(\mathcal{V})$ presents X, is a subset of $\mathcal{V}_{max}(X)$.

Proposition 2. *Let $X \subset \Sigma^{\mathbb{Z}}, \widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be subshifts, and let $\varphi : X \to \widetilde{X}$ be a topological conjugacy. Then*

$$(x, C) \to \varphi(x, C) \quad ((x, C) \in C^\bullet(\mathcal{V}_{max}(X)))$$

is a topological conjugacy of $C^\bullet(\mathcal{V}_{max}(X))$ onto $C^\bullet(\mathcal{V}_{max}(\widetilde{X})$.

Proof. An application of Lemma 1 yields the continuity of the mapping

$$(x, C) \to \varphi(x, C) \quad ((x, C) \in C^\bullet(\mathcal{V}_{max}(X)). \quad \square$$

Given subshifts $X \subset \Sigma^{\mathbb{Z}}, \widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi : X \to \widetilde{X}$, φ will also denote the topological conjugacy that sends the point $(x, C) \in C^\bullet(\mathcal{V}_{max}(X))$ to the point $\varphi(x, C) \in C^\bullet(\mathcal{V}_{max}(\widetilde{X})$. Given transition- and retro-complete subsets $\mathcal{V} \subset \mathcal{V}_{max}(X)$ and $\widetilde{\mathcal{V}} \subset \mathcal{V}_{max}(\widetilde{X})$,

where $\mathcal{G}(\mathcal{V})$ presents X and $\mathcal{G}(\widetilde{\mathcal{V}})$ presents $\widetilde{X}$, such that $\varphi(C^\bullet(\mathcal{V})) = C^\bullet(\widetilde{\mathcal{V}})$, we write also $\varphi(\mathcal{V}) = \widetilde{\mathcal{V}}$ and $\varphi(tM(\mathcal{V})) = tM(\widetilde{\mathcal{V}})$.

Given a construction for a subshift $X \subset \Sigma^{\mathbb{Z}}$ of a transition- and retro-complete subset $\mathcal{V}_X \subset \mathcal{V}_{max}(X)$ such that $\mathcal{G}(\mathcal{V}_X)$ presents X, we say that $\mathcal{V}_X$ is canonical, if for subshifts $X \subset \Sigma^{\mathbb{Z}}, \widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ and a topological conjugacy $\varphi : X \to \widetilde{X}$, $\varphi(\mathcal{V}_X) = \mathcal{V}_{\widetilde{X}}$. $\mathcal{V}_{max}$ itself is canonical by Proposition 2. The standard example of the canonical situation is

$$\mathcal{V}_{standard}(X) = \{\Gamma^+_\infty(x^-) : x^- \in X_{(-\infty,0]}\}.$$

It is

$$tM(\mathcal{V}_{standard}(X)) = \{(x_i, \Gamma^+_\infty((x_j)_{j<i}))_{i\in\mathbb{Z}} : (x_i)_{i\in\mathbb{Z}} \in X\},$$

and

$$C^\bullet(\mathcal{V}_{standard}(X)) = \{(x, W^-(X)) : x \in X\},$$

and from this it is seen that $\mathcal{V}_{standard}$ is canonical. The finiteness of $\mathcal{V}_{standard}$ characterizes the sofic case [W]. That $\mathcal{V}_{standard}$ is canonical was first noted in the sofic case in [Kr1, Kr2]. For a subshift $X \subset \Sigma^{\mathbb{Z}}$ the closure of $\mathcal{V}_{standard}(X)$ is a canonical compact Shannon graph that presents X. This presentation appears in Matsumoto's theory of λ - graph systems (see [M, KM]). Also in non-sofic cases $\mathcal{V}_{standard}(X)$ itself can be compact. For instance, the coded system (see [BH]) with alphabet $\Sigma = \{\gamma, 0, 1\}$ and code $\{\gamma 0^l 0^l : l \in \mathbb{N}\}$ has a compact $\mathcal{V}_{standard}$, while the coded system with alphabet $\Sigma = \{\gamma, 0, 1\}$ and code $\{\gamma 0^{2l} 0^{2l} : l \in \mathbb{N}\}$ has a $\mathcal{V}_{standard}$, that is not compact. The content of the following proposition is that the compactness of $\mathcal{V}_{standard}$ is an invariant of topological conjugacy.

Proposition 3. *Let $X \subset \Sigma^{\mathbb{Z}}$ and $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be topologically conjugate subshifts, and let $\mathcal{V}_{standard}(X)$ be compact. Then $\mathcal{V}_{standard}(\widetilde{X})$ is also compact.*

Proof. This follows from Proposition 2. □

Another example of a canonical presenting Shannon graph is the word Shannon graph of a subshift $X \subset \Sigma^{\mathbb{Z}}$,

$$\mathcal{V}_{word}(X) = \{\{x^+\} : x^+ \in X_{[1,\infty)}\}.$$

It is

$$tM(\mathcal{V}_{word}(X)) = \{(x_i, \{(x_j)_{j\geq i}\})_{i\in\mathbb{Z}} : (x_i)_{i\in\mathbb{Z}} \in X\},$$

and

$$C^\bullet(\mathcal{V}_{word}(X)) = \{(x, \{x\}) : x \in X\}.$$

$\mathcal{V}_{word}$ is compact.

2. Notions of Synchronization

We describe synchronizing shifts and their synchronizing Shannon graphs and then consider the more general notions of s-synchronization and a-synchronization.

2.1. **Synchronization.** A word b that is admissible for a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ is called a synchronizing word of X, if for $c \in \Gamma^-(b), d \in \Gamma^+(b)$ one has that $cbd \in \mathcal{L}(X)$. Equivalently, a synchronizing word of X can be defined as a word $b \in \mathcal{L}(X)$ such that $\Gamma^-(b) = \omega^-(b)$, or, such that $\Gamma^+(b) = \omega^+(b)$. We denote the set of synchronizing words of a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ by $\mathcal{L}_{synchro}(X)$.

Lemma 3. *Let $X \subset \Sigma^{\mathbb{Z}}, \widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be topologically transitive subshifts and let $L \in \mathbb{Z}_+$ be such that there is a topological conjugacy of X onto $\widetilde{X}$ that has together with its inverse the coding window $[-L, L]$, with its inverse given by a block map*

$$\widetilde{\Phi} : \widetilde{X}_{[-L,L]} \to \Sigma.$$

Let

$$b \in \mathcal{L}_{synchro}(X),$$

and let $\widetilde{b} \in \mathcal{L}(\widetilde{X})$ be a word such that $b = \widetilde{\Phi}(\widetilde{b})$. Then

$$\widetilde{b} \in \mathcal{L}_{synchro}(\widetilde{X}),$$

Proof. Let $\widetilde{c} \in \Gamma^-(\widetilde{b}), \widetilde{d} \in \Gamma^+(\widetilde{b})$ and choose words $\widetilde{c}' \in \Gamma^-(\widetilde{c}\widetilde{b})$ and $\widetilde{d}' \in \Gamma^+(\widetilde{b}\widetilde{d})$ of length $2L$. With words $c \in \Gamma^-(b), d \in \Gamma^+(b)$, that are given by

$$\widetilde{\Phi}(\widetilde{c}'\widetilde{c}\widetilde{b}) = cb, \quad \widetilde{\Phi}(\widetilde{b}\widetilde{d}\widetilde{c}'\ ) = bd,$$

one has

$$cbd \in \mathcal{L}(X),$$

and

$$\widetilde{c}\widetilde{b}\widetilde{d} = \Phi(cbd). \quad \Box$$

A topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ that has a synchronizing word is called synchronizing (see [BH]). For $a \in \mathcal{L}_{synchro}(X)$ and $b \in \Gamma^+(a)$ also $ab \in \mathcal{L}_{synchro}(X)$. It follows for a synchronizing subshift X that the set

$$\mathcal{V}_{synchro}(X) = \{\Gamma^+_\infty(a) : a \in \mathcal{L}_{synchro}(X)\}$$

is the vertex set of an irreducible Shannon sub-graph $\mathcal{G}(\mathcal{V}_{synchro}(X))$ of $\mathcal{G}(\mathcal{V}_{max}(X))$, that we call the synchronizing Shannon graph of X.

$\mathcal{G}(\mathcal{V}_{synchro}(X))$ presents X. Topologically transitive sofic systems can be characterized as the synchronizing subshifts whose synchronizing Shannon graph is finite [W]. The Shannon graph $\mathcal{G}(\mathcal{V}_{synchro}(X))$ was first constructed in the sofic case in [F], where the term "Shannon graph" was introduced. It follows from Lemma 3 that synchronization is an invariant of topological conjugacy. The presentation of a synchronizing subshift by its synchronizing Shannon graph is canonical. This is the content of the following theorem.

Theorem 1. *Let $X \subset \Sigma^{\mathbb{Z}}, \widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be synchronizing subshifts, and let $\varphi : X \to \widetilde{X}$ be a topological conjugacy. Then*

$$\varphi(\mathcal{V}_{synchro}(X)) = \mathcal{V}_{synchro}(\widetilde{X}).$$

Proof. Let $L \in \mathbb{Z}_+$ be such that $[-L, L]$ is a coding window for φ and for φ^{-1}.

Let

$$(x, C) \in C^\bullet(\mathcal{V}_{synchro}(X)).$$

and let $j \in \mathbb{Z}$. Let b be a synchronizing word of X such that

$$V_{-3L}(C) = \Gamma^+_\infty(b), \tag{2.1}$$

end let $d \in \Gamma^+(b)$ be such that $bd \in \Gamma^-(b)$. Consider the point $x' \in X$ such that $X_{(-\infty,-3L)}$ carries the left infinite concatenation of db, and such that

$$x'_{[-3L,\infty)} = x_{[-3L,\infty)}. \tag{2.2}$$

Let $(x', C') \in C^\bullet(\mathcal{V}_{synchro}(X))$ be given by

$$V_i(C') = \Gamma^+_\infty(x'_{(-\infty,i]}), \quad i \in \mathbb{Z},$$

and let

$$(\widetilde{x}', \widetilde{C}') = \varphi(x', C').$$

By Lemma 3

$$(\widetilde{x}', \widetilde{C}') \in C^\bullet(\mathcal{V}_{synchro}(\widetilde{X})),$$

and by Lemma 1 and by (2.1) and (2.2)

$$V_j(\widetilde{C}') = V_j(\widetilde{C}).$$

By symmetry the theorem is proved. $\square$

2.2. **s-synchronization.** A word b that is admissible for a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ is called an s-synchronizing word of X, if for all $c \in \mathcal{L}(X)$ there exists a $d \in \Gamma^{+}(c)$ such that $cd \in \omega^{-}(b)$. A synchronizing word is s-synchronizing. We denote the set of s-synchronizing words of a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ by $\mathcal{L}_{s-synchro}(X)$. We note that for an s-synchronizing word b there exists in particular a word $d \in \Gamma^{+}(b)$ such that $bd \in \omega^{-}(b)$ and this implies that $\omega^{-}_{\infty}(b) \neq \emptyset$.

Lemma 4. *Let $X \subset \Sigma^{\mathbb{Z}}$ be a topologically transitive subshift, and let*

$$b \in \mathcal{L}_{s-synchro}(X). \tag{2.3}$$

Let $d \in \mathcal{L}(X)$ be such that $bd \in \omega^{-}(b)$ and let l denote the length of the word bd.

Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be a subshift that is topologically conjugate to X, and let $L \in \mathbb{Z}_{+}$ be such that $[-lL, lL]$ is a coding window of a topological conjugacy of X onto $\widetilde{X}$, the topological conjugacy being given by the block map

$$\Phi : X_{[-lL,lL]} \to \widetilde{\Sigma}.$$

Then

$$\widetilde{b} = \Phi(b(db)^{4L}) \in \mathcal{L}_{s-synchro}(\widetilde{X}).$$

Proof. For the proof let $\widetilde{c} \in \mathcal{L}(\widetilde{X})$ and let $c \in \mathcal{L}(X)$ be such that

$$\widetilde{c} = \Phi(c).$$

By (2.3) there is a $d' \in \Gamma^{+}(c)$ such that

$$cd' \in \omega^{-}(b). \tag{2.4}$$

Let $\widetilde{d} \in \Gamma^{+}(\widetilde{c})$ be given by

$$\widetilde{c}\widetilde{d}\widetilde{b} = \Phi(cd'b(db)^{4L}).$$

(2.4) implies that

$$\widetilde{c}\widetilde{d} \in \omega^{-}(\widetilde{b}). \quad \square$$

A topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ with an s-synchronizing word is called s-synchronizing (see [Kr3]). For $b \in \mathcal{L}_{s-synchro}(X)$ and $a \in \Gamma^{+}(b)$ also $ba \in \mathcal{L}_{s-synchro}(X)$. It follows for an s-synchronizing subshift X that the set

$$\mathcal{V}_{s-synchro}(X) = \{\Gamma^{+}_{\infty}(b) : b \in \mathcal{L}_{s-synchro}(X)\}$$

is the vertex set of an irreducible countable Shannon sub-graph of $\mathcal{G}(\mathcal{V}_{max}(X))$, that we denote by $\mathcal{G}(\mathcal{V}_{s-synchro}(X))$, and that we call the s-synchronizing Shannon graph of X. $\mathcal{G}(\mathcal{V}_{s-synchro}(X))$ presents

X. It follows from Lemma 4 that s-synchronization is an invariant of topological conjugacy. The presentation of an s-synchronizing shift by its s-synchronizing Shannon graph is canonical. This is the content of the following theorem, that is shown in the same way as Theorem 1, Lemma 4 taking the place of Lemma 3.

Theorem 2. *Let $X \subset \Sigma^{(\mathbb{Z})}, \widetilde{X} \subset \widetilde{\Sigma}^{(\mathbb{Z})}$ be s-synchronizing subshifts, and let $\varphi : X \to \widetilde{X}$ be a topological conjugacy. Then*

$$\varphi(\mathcal{V}_{s-synchro}(X)) = \mathcal{V}_{s-synchro}(\widetilde{X}).$$

2.3. **a-synchronization.** For a subshift $X \subset \Sigma^{\mathbb{Z}}$, we denote for $b \in \mathcal{L}(X)$ by $\omega^+_\circ(b)$ the set of words $c \in \omega^+(b)$ that appear as prefixes of sequences in $\omega^+_\infty(b)$,

$$\omega^+_\circ(b) = \{x^+_{[1,n]} : x^+ \in \omega^+_\infty(b), n \in \mathbb{N}\}.$$

A word b that is admissible for a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ is called an a-synchronizing word for X if b satisfies the following two conditions a-s(1) and a-s(2):

Condition a-s(1). *For $c \in \mathcal{L}(X)$ there exists a $d \in \Gamma^-(c)$ such that $dc \in \omega^+(b)$.*

Condition a-s(2). *For $c \in \omega^+_\circ(b)$ there exists a $d \in \Gamma^+(c)$ such that $cd \in \Gamma^-(b)$ and such that $cdb \in \omega^+(b)$ and $\omega^+_\infty(bcdb) = \omega^+_\infty(b)$.*

Condition a-s(1) is equivalent to requiring that the word b is an s-synchronizing word for the inverse of the subshift $X \subset \Sigma^{\mathbb{Z}}$. We denote the set of a-synchronizing words of a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ by $\mathcal{L}_{a-synchro}(X)$.

Lemma 5. *Let b be an a-synchronizing word of the topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$. Then there exists a word $d \in \Gamma^-(b)$ such that $db \in \omega^+(b)$ and $\omega^+_\infty(bdb) = \omega^+_\infty(b)$.*

Proof. First choose by a-s(1) a word $d' \in \Gamma^-(b)$ such that $d'b \in \omega^+(b)$. Then choose by a-s(2) a word $d'' \in \Gamma^+(d'b) \cap \Gamma^-(b)$ such that $d'bd''b \in \omega^+(b)$ and such that $\omega^+_\infty(bd'bd''b) = \omega^+_\infty(b)$. Set $d = d'bd''$. ☐

From the preceding lemma it is seen that the formulation of the following lemma is meaningful.

Lemma 6. *Let $X \subset \Sigma^{(\mathbb{Z})}$ be a topologically transitive subshift, and let*

$$b \in \mathcal{L}_{a-synchro}(X). \tag{2.5}$$

Let $d \in \mathcal{L}(X)$ be such that $db \in \omega^+(b)$ and $\omega^+_\infty(bdb) = \omega^+_\infty(b)$, and let l denote the length of the word db.

Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be a subshift that is topologically conjugate to X, and let $L \in \mathbb{Z}_+$ be such that $[-lL, lL]$ is a coding window of a topological conjugacy of X onto $\widetilde{X}$, the topological conjugacy being given by the block map

$$\Phi : X_{[-lL,lL]} \to \widetilde{\Sigma}.$$

Then

$$\widetilde{b} = \Phi((bd)^{4L}b) \in \mathcal{L}_{a-synchro}(\widetilde{X}).$$

Proof. That $\widetilde{b}$ satisfies a-s(1) is Lemma 4.

For the proof that $\widetilde{b}$ also satisfies a-s(2), let $\widetilde{c} \in \omega^+_\circ(\widetilde{b})$, and denote the length of the word $\widetilde{c}$ by m. Choose an $\widetilde{x}^+ \in \omega^+_\infty(\widetilde{b})$ such that $\widetilde{x}^+_{[1,m]} = \widetilde{c}$, and let $x^+ \in \omega^+_\infty(b)$ be given by

$$\Phi((bd)^{2l}bx^+) = \widetilde{b}\widetilde{x}^+.$$

Let

$$c = x^+_{[1,m+lL]}.$$

By (2.5), and according to a-s(2), there is a $d' \in \Gamma^+(c) \cap \Gamma^-(b)$ such that $cd'b \in \omega^+_\infty(b)$, and $\omega^+_\infty(bcd'b) = \omega^+_\infty(b)$. A $\widetilde{d} \in \Gamma^+(\widetilde{c} \cap \Gamma^-(\widetilde{b})$ such that

$$\widetilde{c}\widetilde{d}\widetilde{b} \in \omega^+_\infty(\widetilde{b}),$$

is given by

$$\Phi((bd)^{2L}cd'b(db)^{4L}) = \widetilde{b}\widetilde{c}\widetilde{d}\widetilde{b}.$$

To show that also

$$\omega^+_\infty(\widetilde{b}\widetilde{c}\widetilde{d}\widetilde{b}) \subset \omega^+_\infty(\widetilde{b}),$$

let

$$\widetilde{y}^+ \in \omega^+_\infty(\widetilde{b}\widetilde{c}\widetilde{d}\widetilde{b}). \tag{2.6}$$

There is a

$$y^+ \in \Gamma^+_\infty((bd)^{2L}bcd'b(db)^{3L})$$

given by

$$\Phi(b(db)^{3L}y^+) = \widetilde{b}\widetilde{c}\widetilde{d}\widetilde{b}\widetilde{y}^+.$$

(2.6) implies that

$$y^+ \in \omega^+_\infty((bd)^{2L}bcd'b(db)^{3L}),$$

which implies that

$$y^+ \in \omega^+_\infty(b),$$

which then implies that

$$\widetilde{y}^+ \in \omega^+_\infty(\widetilde{b}). \quad \square$$

For an a-synchronizing word b of the topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ and for a word $a \in \omega^+_\circ(b)$ the word ba is again a-synchronizing for X. It follows that the set

$$\mathcal{V}_{a-synchro}(X) = \{\omega^+_\infty(b) : b \in \mathcal{L}_{a-synchro}(X)\}$$

is the vertex set of a Shannon sub-graph of $\mathcal{G}(\mathcal{V}_{max}(X))$, that we call the a-synchronizing Shannon graph of X, and that we denote by $\mathcal{G}_{a-synchro}(X)$. $\mathcal{G}_{a-synchro}(X)$ is the union of its irreducible components all of which present X. One can see from condition a.s(2) and from Lemma 4 that the number of irreducible component of $\mathcal{G}_{a-synchro}(X)$ is an invariant of topological conjugacy. In an attempt to maintain an anology with synchronizaion and s-synchronization we say that a topologically transitive subshift $X \subset \Sigma^{\mathbb{Z}}$ is a-synchronizing, if X has a a-synchronizing word and if $\mathcal{G}_{a-synchro}(X)$ is irreducible (comp. [Kr3]). The presentation of an a-synchronizing subhift by its a-synchronizing Shannon graph is canonical. This is the content of the following theorem, that is shown in the same way as Theorem 1, Lemma 6 taking the place of Lemma 3.

Theorem 3. *Let $X \subset \Sigma^{\mathbb{Z}}, \widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ be a-synchronizing subshifts, and let $\varphi : X \to \widetilde{X}$ be a topological conjugacy. Then*

$$\varphi(\mathcal{V}_{a-synchro}(X)) = \mathcal{V}_{a-synchro}(\widetilde{X}).$$

References

[BP] M.-P. Béal, D. Perrin, *Symbolic Dynamics and Finite Automata*, Handbook of Formal Languages Vol.2, G.Rozenberg, A.Salamaa Eds. Springer (1997), 463–506.

[BH] F. Blanchard, G. Hansel, *Systèmes codés*, Theoret. Comput. Sci. **44** (1986), 17–49.

[F] R. Fischer, *Sofic systems and graphs*, Monatsh. Math. **80** (1975), 179–86.

[Ki] B. P. Kitchens, *Symbolic dynamics*, Springer, Berlin, Heidelberg, New York (1998).

[Kr1] W. Krieger, *On sofic systems I*, Israel J. Math. **48** (1984), 305 – 330.

[Kr2] W. Krieger, *On sofic systems II*, Israel J. Math. **4** (1987), 167 – 176.

[Kr3] W. Krieger, *On subshifts and topological Markov chains*, Numbers, information and complexity (Bielefeld 1998), Kluwer Acad. Publ. Boston MA. (1987), 453 – 472.

[KM] W. Krieger and K. Matsumoto, *Shannon graphs, subshifts and lambda-graph systems*, J. Math. Soc. Japan **4** (2002), 877 – 899.

[LM] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge. (1995).

[M] K. Matsumoto, *Presentations of subshifts and their topological conjugacy invariants*, Doc. Math. **4** (1999), 285 – 340.

[W] B. Weiss, *Subshifts of finite type and sofic systems*, Monats. Math. **77** (1973), 462 – 474.

Institut für Angewandte Mathematik
Universität Heidelberg
Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
e-mail: krieger@math.uni-heidelberg.de